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Random Pinning of Localized States and the Birth of Deterministic Disorder within Gradient Models

K. A. Gorshkov,[1] L. N. Korzinov,[1] M. I. Rabinovich,[1] and L. S. Tsimring[1,2]

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The birth of spatial disorder from almost regular initial conditions within the Swift–Hohenberg model equation with subcritical bifurcation is considered. The complexity of the space series (measured by the spatial K_2-entropy) grows with time and reaches a stationary value depending on the period of the initial regular disturbance. A qualitative model is suggested describing the process via the birth of localized structures and its subsequent disordering due to weak interaction.

KEY WORDS: Random pinning; gradient models; localized structures; spatiotemporal chaos.

1. INTRODUCTION

The mechanisms and the scenarios according to which the smooth and regular picture of a nonlinear field evolves into a spatially disordered, chaotic pattern as $t \to \infty$ is one of the most fascinating and essential problems in the nonlinear dynamics of nonequilibrium media. Today the existence of spatial disorder within deterministic field models, at least within one-dimensional models, is self-evident, but the scenarios of its birth in the evolution of the field need further investigation.[1,2] In this paper we will describe the birth of spatial disorder through the appearance and interaction of localized states of the field, which seems to be a generic mechanism. We will present results of computer experiment and qualitative theory demonstrating that the evolution along the sequence "a smooth,

[1] Institute of Applied Physics, Russian Academy of Science, Nizhni Novgorod, 603600 Russia.
[2] Present address: Institute for Nonlinear Science, University of California at San Diego, La Jolla, California 92093-0402.

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regular field with small initial irregularity → a lattice of localized states → spatial chaos" is indeed realized within a gradient model

$$\frac{\partial u}{\partial t} = -u + \beta u^2 - u^3 - (\kappa_0^2 + \nabla^2)^2 u \tag{1}$$

This is a generalized Swift–Hohenberg equation that describes subcritical bifurcation in pattern-forming systems. The parameter β determines the instability threshold (only disturbances of finite amplitude $u > u_0$ grow, where $u_0 = \beta/2 - [\beta^2/4 - (1 + \kappa_0^4)]^{1/2}$ and κ_0 is the inverse characteristic spatial scale of the system).

In the present paper we will numerically investigate (1) and its one-dimensional version:

$$\frac{\partial u}{\partial t} = -u + \beta u^2 - u^3 - \left(\kappa_0^2 + \frac{\partial^2}{\partial x^2}\right)^2 u \tag{2}$$

We start with a nearly harmonic initial field distribution and show that it evolves into a chain of localized pulses that later move apart, due to instability, at a random distance from one another and stick at potential minima forming, as $t \to \infty$, a static disordered lattice. For quantitative characterization of the "chaoticity" of a one-dimensional space series we computed the K_2-entropy. The entropy of the final state of the lattice depends on the period of the initial field distribution. If the initial distribution has different periods in different portions, then, as $t \to \infty$, an inhomogeneous static regime with "different" disorder is established for which a piecewise constant dependence of the entropy on coordinate is typical. Note that the mechanism responsible for the existence of spatial chaos due to random pinning of defects in the oscillating potential of one another was proposed within a model with supercritical bifurcation by Coullet *et al.*[3] The mechanism of the birth of disorder in the one-dimensional generalized Swift–Hohenberg equation can be studied on a simple model treating the field distribution as a chain of weakly interacting "particles" with the specific potential of interaction. We also performed a numerical simulation of the two-dimensional equation (1) and also found the onset of a disordered two-dimensional pattern. However, the question of the quantitative description of chaos in more than one dimension is more complicated (see, e.g., ref. 1) and we do not discuss it in the present paper.

2. COMPUTER EXPERIMENT

We are interested in the time evolution of the spatial distribution of the "field" that is described by Eq. (1), whose solutions are determined in a rather broad region Ω, so that there exists a free energy functional

$$F=\int_{\Omega}\left\{\frac{1}{2}u^2-\frac{\beta}{3}u^3+\frac{u^4}{4}+\frac{1}{2}\left[(\kappa_0^2+\nabla^2)u\right]^2\right\}d\mathbf{r} \tag{3}$$

Equation (1) in this case may be represented in a gradient form

$$\frac{\partial u}{\partial t}=-\frac{\delta F}{\delta u} \tag{4}$$

Only static attractors may exist in the phase space of this system because F may only decrease along the trajectory

$$\frac{dF}{dt}=-\int\left(\frac{\partial u}{\partial t}\right)^2 d\mathbf{r}\leqslant 0$$

Periodic, quasiperiodic, or chaotic spatial distributions may correspond to these attractors. The number of attractors may, in principle, be arbitrarily great. Indeed, according to (1) and (4), all static solutions that are established as $t\to\infty$ meet the system $\delta F/\delta u=0$ or $(\kappa_0^2+\nabla^2)^2 u-f(u)=0$, where $f(u)=\beta u^2-u^3-u$.

Representing the initial periodic state of the field as a point in the phase space of the gradient system of interest, we can formulate the problem of the birth of disorder in the form of a simple question: Is this point contained in the attraction basin of the attractor corresponding to the disordered field distribution as $t\to\infty$? Physically, however, this formulation of the problem does not sound natural enough. We must be concerned with a set of close initial field distributions rather than with one particular distribution. An initial phase volume with a characteristic size $\varepsilon\ll 1$ and not a point corresponds to such a set in phase space. Then the question will have a quite different formulation: Will an irregular state of the field, the statistical characteristics of which depend only on the period of the initial distribution and are independent of ε (including $\varepsilon\to 0$), set in as $t\to\infty$? To answer these questions, we performed a number of numerical experiments with the one-dimensional version of Eq. (1), namely, Eq. (2).

Equation (2) was integrated employing a spectral method for $N=4096$ spatial harmonics with periodic boundary conditions $u(0)=u(l)$ for

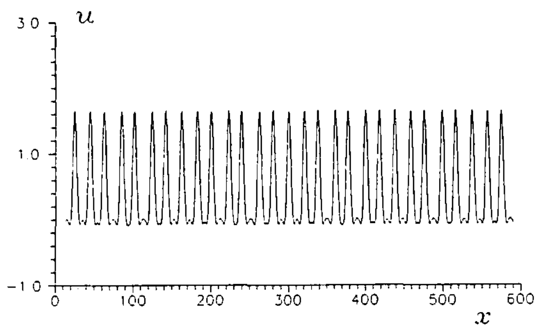


Fig. 1. Static stable field distribution for initial conditions (5) with $a = 1.8$ and $\varepsilon = 0.2$. The picture was taken after a long time ($T = 100$ in dimensionless units) and is actually a field distribution as $T \to \infty$.

$l = 100$, $\kappa_0 = 0.5$, and $\beta = 2.3$.[3] We took a nearly harmonic initial field distribution

$$u(x) = a \sin kx + \varepsilon f(x) \tag{5}$$

where $\varepsilon \ll a$ and $f(x)$ is random function.

The wavenumber k of the harmonic distribution was chosen so that the periodic lattice of localized structures emerging from it when $\varepsilon = 0$ should correspond to the local maximum of free energy in Eq. (1), i.e., so that such a lattice should be unstable relative to small disturbances. The small irregular component $\varepsilon \ll 1$ acts in this case only as an "embryo" to initiate instability. Then, a solution like the one shown in Fig. 1 is established as $t \to \infty$. The parameters for initial conditions (5) were taken to be $a = 1.8$ and $\varepsilon = 0.2$.

Numerical experiment illustrates the birth of spatial chaos whose properties are universal in that they do not depend on statistical characteristics of $f(x)$, but, instead, are determined by the dynamical system itself and by the initial regular pattern.

We computed a K_2-entropy using the Grassberger–Procaccia algorithm.[4] At the first step a phase space is reconstructed via "space-delay" embedding:

$$\mathbf{u} = \{u(x, t), u(x + X, t), \ldots, u(x + (d_E - 1)\, X, t)\}$$

[3] We consider a "large-box" dissipative system; therefore the concrete form of the boundary conditions is of no significance.

where X is a space delay determined by the mutual information criterion[5] and d_E is the embedding dimension. Then the correlation integrals $C(r; d_E)$

$$C(r; d_E) = \lim_{N \to \infty} \frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{j=1}^{N} \theta(r - \|\mathbf{u}_i - \mathbf{u}_j\|)$$

are computed for a number of d_E [$\theta(x)$ is the Heaviside function]. The K_2-entropy is found from the formula[4]

$$K_2 = \lim_{r \to 0} \lim_{d_E \to \infty} X^{-1}[C(r, d_E) - C(r; d_E - 1)] \qquad (6)$$

In practice we deal with a finite number N of finite-precision data points, so we have computed K_2 for intermediate values of r and d_E within intervals of scaling. The time dependence of the K_2-entropy calculated for the space series is shown in Fig. 2. Several (5–10) space series corresponding to different realizations of $f(x)$ were usually processed to provide better accuracy of computations. They can be considered as pieces of one infinite space series. The increase of entropy and, eventually, its asymptotic form indicate that static disorder occurs in (2).

Numerical experiment demonstrated that finite-dimensional disorder is formed in two stages (Fig. 3). At the first (fast) stage of evolution, the initial field distribution transforms, in time $t \lesssim 10.0$, into a chain of solitons that are slightly shifted (for small ε) with respect to the equidistant arrangement. The field structure of such a solution and its tails are shown

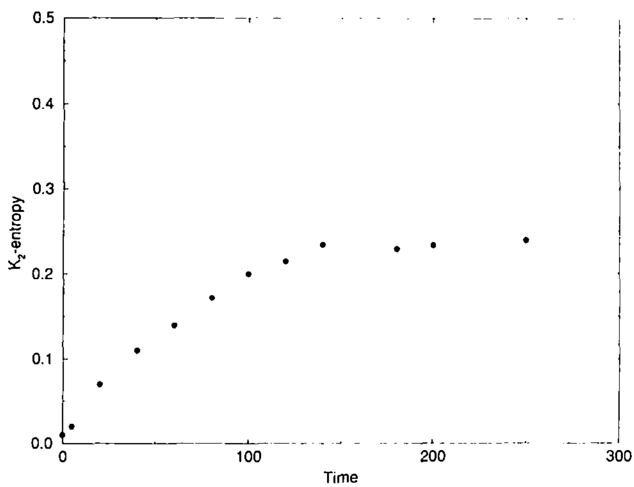


Fig. 2. K_2-entropy as a function of time within model (2) with initial conditions (5).

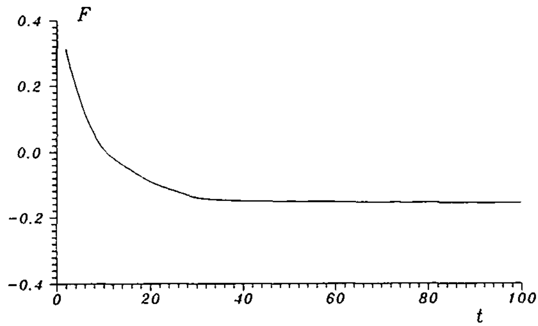


Fig. 3. The variation of free energy functional F under initial conditions of the form (5). It is seen that $F(t)$ has two characteristic portions: (1) fast decrease of F corresponding to "particle" formation for $0 \lesssim t \gtrsim 10.0$, and (2) slow variation of F, for $T \gtrsim 15.0$, when the "solitons" formed at stage 1 move apart at a distance such that each soliton is at the potential minimum of the others (stable static field distribution).

in Fig. 4. If we have a nearly periodic initial distribution, the solution lattice is nearly periodic, too. The medium remembers the initial conditions at this stage of evolution. As t increases ($t \gtrsim 15.0$), the solitons slowly shift relative to one another and the initial conditions are no longer remembered if the lattice is unstable.

3. QUALITATIVE ANALYSIS

Results of computer experiment on the transformation of an almost regular initial field distribution into a random sequence of solitons may be

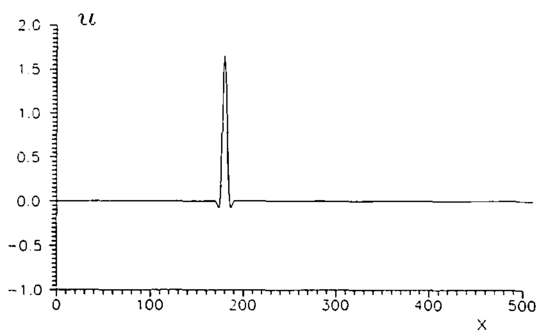


Fig. 4. Field amplitude for the solution in the form of a single soliton (localized state).

interpreted pictorially within the following model. For not too short distances between the neighboring solitons (of the order of their characteristic size or larger), the asymptotic method[6] enables us to describe the soliton motion as the dynamics of a chain of particles with specific interaction potential that is determined by the structure of the "tails" of neighboring localized particles (Fig. 4),[4]

$$M \frac{dx_i}{dt} = f_i \equiv \frac{\partial}{\partial x_i} u_i \tag{7}$$

Here x_i is the coordinate of the center of the ith soliton, $M = \int (u_x^{(0)})^2 \, dx$ is the mobility of the soliton having the structure $u^{(0)}(x)$, and u_i is the potential produced by two neighboring solitons $i \pm 1$ at the point x_i:

$$\begin{aligned} u_i = U_0(e^{-\nu|x_i - x_{i-1}|} \cos(\mu\, |x_i - x_{i-1}| + \phi_0) \\ + e^{-\nu|x_i - x_{i+1}|} \cos(\mu\, |x_i - x_{i+1}| + \phi_0)) \end{aligned} \tag{8}$$

where $\nu = \mathrm{Re}(i - \kappa_0^2)^{1/2}$, $\mu = \mathrm{Im}(i - \kappa_0^2)^{1/2}$, and U_0 and ϕ_0 are numerical constants to be determined from simulations. For our parameters $\kappa_0 = 0.5$ and $\beta = 2.3$ we obtained $\nu = 0.62$, $\mu = 0.80$, $\phi = -0.08$, $U_0 = 7.15$, and $\phi_0 = -2.51$.

Clearly, an equidistant lattice of "particles" with arbitrary period l ($x_i = il$) corresponds to a stationary state of (7). To determine the stability of that solution, consider a linearized version of (7) for small deviations from a stationary state $\tilde{x}_i = x_i - il$:

$$\dot{\tilde{x}}_i = \sum_j a_{ij} \tilde{x}_j \tag{9}$$

where $a_{ij} \equiv \partial f_i / \partial x_j |_{x_i = jl}$ and dot denotes the differentiation over new time $t_n = U_0 t / M$. It is easy to check that $a_{ij} = 0$ for $|i - j| > 1$,

$$a_{ij} = -A = -e^{\nu l} \sin(\mu l + \phi_0 + \phi)(1 + \kappa_0^4)^{1/2} \tag{10}$$

for $|i - j| \neq 1$, and

$$a_{ii} = 2A$$

[4] The force of interaction in gradient systems determines the velocity of localized states rather than their acceleration. We take into account only neighboring structures, since the interaction with others is exponentially small by comparison with the first ones and lies beyond the accuracy of the first-order asymptotic analysis.

Here $\phi = \arctan(\nu^2 - \mu^2)/2\nu\mu$. The eigenvalues for the three-diagonal matrix

$$\begin{pmatrix} 2A & -A & 0 & \cdots & & 0 & \\ -A & 2A & -A & \cdots & & & \\ & & & \cdots & & & \\ & 0 & & \cdots & 0 & -A & 2A \end{pmatrix} \tag{11}$$

are known[7]

$$\lambda_k = 2A\left(1 + \cos\frac{\pi k}{N+1}\right), \qquad k = 1,..., N \tag{12}$$

where N is the order of the matrix. Therefore, we can conclude that the stability of the equidistant chain of structures is determined by the sign of A. The lattice is unstable for periods l inside the intervals

$$2n\pi < \mu l + \phi_0 + \phi < (2n+1)\,\pi, \qquad n = 0, 1, 2,... \tag{13}$$

This is only an approximate formula for the regions of instability, since we derived it through asymptotic analysis of soliton interaction taking into account only two nearest neighbors and the approximate form of the potential (10). However, numerical experiments carried out with the model (2) showed that (13) gives a reasonable estimation for the periods of initial condition leading to an unstable lattice of solitons. Using the formula (13), we can estimate the period of intervals of "unstable" l as $L = 2\pi/\mu = 7.85$. Numerical data for the K_2-entropy give $L \simeq 6.5$. Now the question arises, What happens if the period of initial sinusoidal disturbance is within one of the instability ranges? Two qualitatively different possibilities can take place here, depending on the initial conditions: (i) periodic and quasi-periodic sequences of "particles"; (ii) chaotic sequences of "particles."

Direct numerical experiment on model (8)–(10) with nearly periodic boundary conditions verifies that spatial K_2-entropy grows with time (Fig. 5). This corresponds to chaotization of solutions of the form (5) within the basic model (2).

It follows from an analysis of system (2) that different unstable soliton lattices and, consequently, different established distributions must correspond to initial distributions with different periods—the effect of multistability. Because of instability, the neighboring solitons may either move apart by "one minimum" (one oscillation of a tail) or move toward one another by the same "unit length." Then it appears obvious that the degree of disorder may only decrease with increasing period of initial distribution! This guess was proved in a computer experiment (Fig. 6): the K_2-entropy actually drops and tends to zero as l grows.

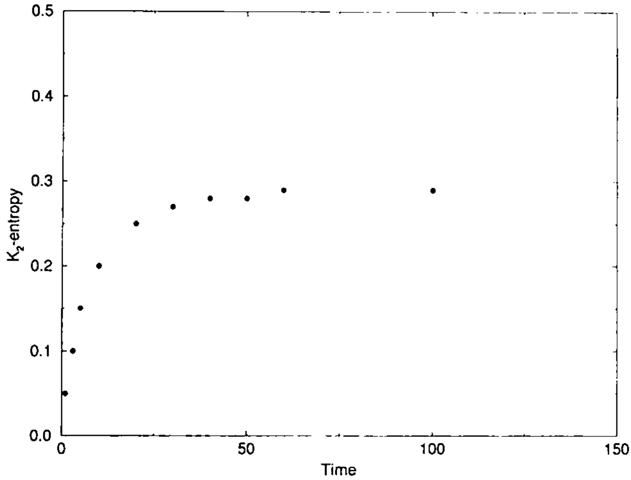


Fig. 5. K_2-entropy as a function of time within model (9)–(9).

4. DISCUSSION

Our results on the evolutionary behavior of finite-dimensional disorder and the onset of disorder with different K_2 depending on periodic initial conditions (Fig. 6) indicate that finite-dimensional disorder may, in principle, arise in different portions of space and have different character-

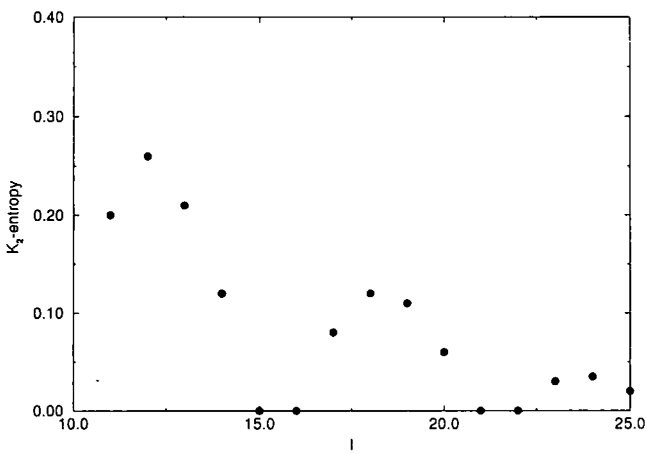


Fig. 6. K_2-entropy versus the period of the initial distribution for the discrete model (8)–(9).

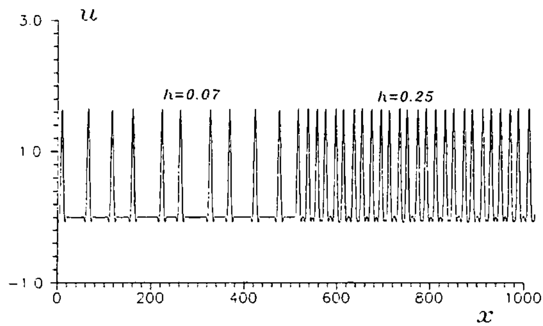


Fig. 7. The onset of different types of disorder in different regions of space at different periods of the initial distribution.

istics. In other words, domains with an irregular location of solitons that differ in the values of K_2 ($K_2 = 0$ is a possible particular case) may coexist in space. The boundaries of these domains correspond to "envelope" defects against the background of finite-dimensional disorder. Results of computer experiment that confirm this concept are presented in Fig. 7, where domains of different finite-dimensional disorder are established under piecewise periodic initial conditions (the corresponding values of entropy are given in the figure).

The same scenario of the birth of spatial chaos via the formation of localized structures and their subsecuent disordering which we illustrated

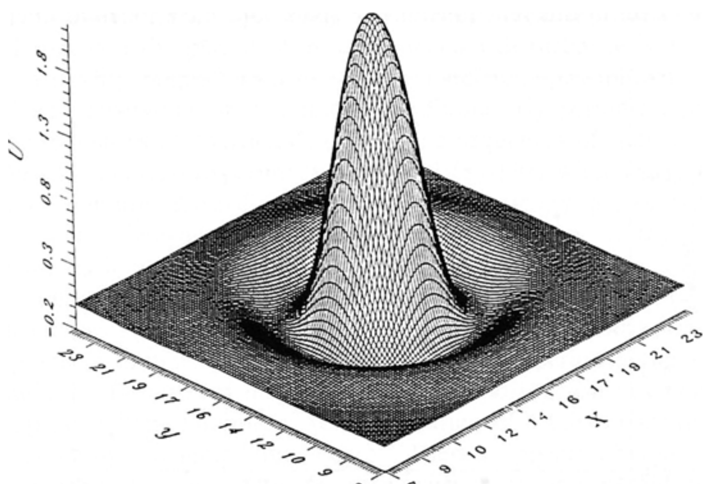
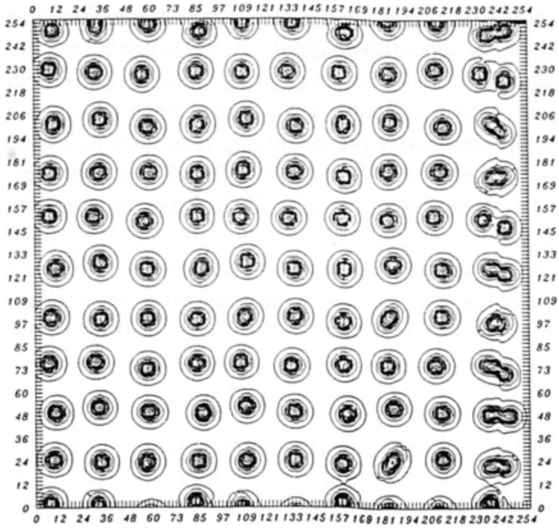
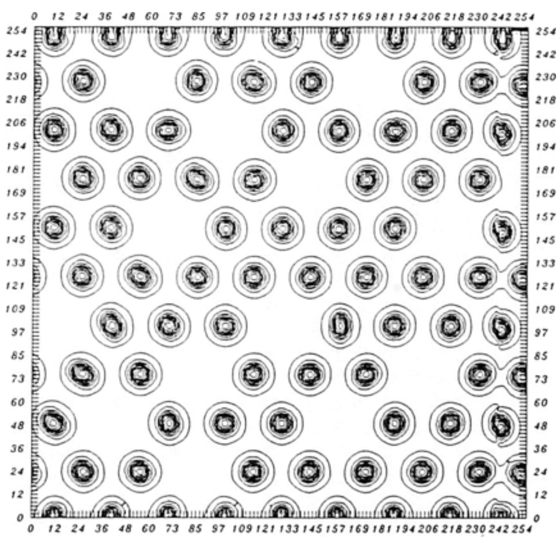


Fig. 8. Localized solitonlike solution of the two-dimensional Swift–Hohenberg equation with subcritical bifurcation (1).



A



B

Fig. 9. Two different two-dimensional disordered states: (A) disordered square lattice; (B) disordered hexagon lattice.

within the framework of the one-dimensional Swift–Hohenberg equation (2) is observed in the two-dimensional Swift–Hohenberg equation (1) with subcritical bifurcation. Indeed, for (1) one can easily find localized structures (see Fig. 8) with oscillating tails. They can form an ideal crystalline lattice and also (if the initial period of the initial disturbance corresponds to an unstable lattice) a number of disordered patterns (see, e.g., Fig. 9). Domains of different disorder can also be observed in the two-dimensional case (Fig. 10).

Stationary irregular solutions of the gradient model (1) are, at the same time, stationary solutions of its conservative (Hamiltonian) analog:

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\delta F}{\delta u} \tag{14}$$

However, while the stability of a disordered chain within (1) follows from the existence of a local minimum of the functional F in the solution of interest, the stability of such a chain within (14) is a more intriguing and complicated problem. It is reminiscent of the problem of the excitation spectrum in the one-dimensional model of a Krönig–Penni liquid (see, e.g., ref. 8). This analogy (including the phenomenon of localization of eigenfunctions) may be extremely fruitful if collective excitations of the chain

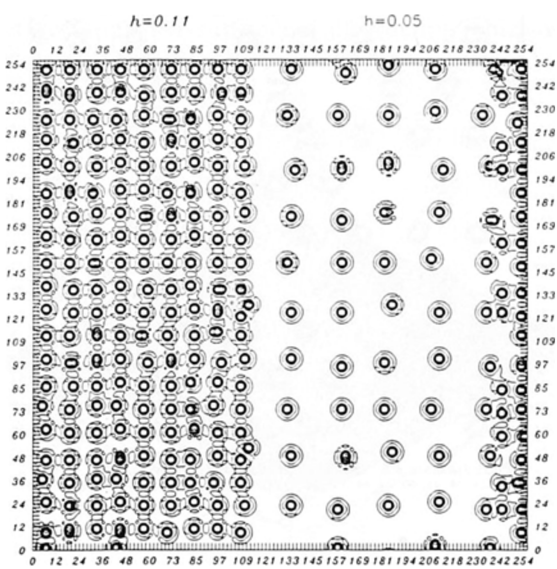


Fig. 10. Coexistence of two different disordered states in space in two dimensions.

and eigenmodes of localized structures are out of resonance. If they *are* in resonance and the resonance is significant, developing instability may transform the system into a quite different state, perhaps without localized structures.

In conclusion, we have shown how the spatially disordered state is formed from almost regular initial conditions within the one- and two-dimensional Swift–Hohenberg equations with subcritical bifurcation. Depending on the period of the initial condition, the disorder may or may not appear. We computed the temporal evolution of the spatial K_2-entropy of the space series which characterizes the complexity of the disordered state and showed that it grows in time and eventually reaches a stationary level. We suggested a simple qualitative model which describes the onset of disorder as the birth and random pinning of localized states with oscillating far-field potential.

It should be noted that, of course, the models under consideration (1) and (2) are rather specific in the sense that they possess the free energy functional (3) and allow for particular localized solutions with oscillation tails. There must be a large variety of systems and models where the spatial disorder appears due to different physical mechanisms than the instability of a periodic chain of "particles" we demonstrated here.

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